HEARTS OF COTORSION PAIRS ARE FUNCTOR CATEGORIES OVER COHEARTS

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Abstract. We study hearts of cotorsion pairs in triangulated and exact categories.We give a sufficient and necessary condition when the hearts have enough projectives. We also show in such condition they are equivalent to functor categories over cohearts of the cotorsion pairs.

1. Introduction

The notion of cotorsion pair in triangulated and exact categories is a general framework to study important structures in representation theory. Recently the notion of hearts of cotorsion pairs was introduced in [N] and [L], and they are proved to be abelian categories, which were known for the heart of t-structure [BBD] and the quotient category by cluster tilting subcategory [KZ, DL]. We refer to [L2] and [AN] for more results on hearts of cotorsion pairs.

In this paper, we give an equivalence between hearts and the functor categories over cohearts. For the details of functor category, see [IY, Definition 2.9]. Throughout this paper, let k be a field.

For any cotorsion pair $(\mathcal{U},\mathcal{V})$ on a triangulated category $\mathcal{T}$, we introduce the notion of *cohearts* of a cotorsion pair, denote by

$$\mathcal{C} = \mathcal{U}[-1] \cap {}^{\perp}\mathcal{U}$$

where ${}^{\perp}\mathcal{U} = \{T \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(T,\mathcal{U}) = 0\}$. This is a generalization of coheart of a co-t-structure. We have the following theorem in triangulated category.

Theorem 1.1. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on a Krull-Schmidt, Hom-finite and k-linear triangulated category $\mathcal{T}$ and H be the associated cohomological functor.The heart of $(\mathcal{U},\mathcal{V})$ has enough projectives $H(\mathcal{C})$ if and only if $(\mathcal{C},\mathcal{K})$ is a torsion pair where $\mathcal{K} = \mathrm{add}(\mathcal{U} * \mathcal{V})$ is the kernel of H. Moreover, when the heart has enough projectives $H(\mathcal{C})$, it is equivalent to the functor category* $\mathrm{mod}\,\mathcal{C}$.

This generalizes [BR, Theorem 3.4] which is for t-structure. One standard example of this theorem is the following: let A be a Noetherian ring with finite global dimension, then the standard t-structure of $\mathsf{D}^b(\mathrm{mod}\,A)$ has a heart $\mathrm{mod}\,A$ with co-heart $\mathrm{proj}\,A$, and we have an equivalence $\mathrm{mod}\,A \simeq \mathrm{mod}(\mathrm{proj}\,A)$ in this case.

For any cotorsion pair $(\mathcal{U},\mathcal{V})$ on an exact category $\mathcal{E}$, we denote

$$\mathcal{C} = \mathcal{U} \cap {}^{\perp_1}\mathcal{U}$$

the *coheart* of $(\mathcal{U},\mathcal{V})$ where ${}^{\perp_1}\mathcal{U} = \{B \in \mathcal{E} \mid \mathrm{Ext}^1_{\mathcal{E}}(B,\mathcal{U}) = 0\}$. Let $\Omega\mathcal{C} = \{X \in \mathcal{E} \mid \text{X admits } 0 \to X \to P \to C \to 0 \text{ where } P \in \mathcal{P} \text{ and } C \in \mathcal{C}\}$, we have the following theorem in exact category.

Theorem 1.2. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on a Krull-Schmidt, Hom-finite and k-linear exact category $\mathcal{E}$ with enough projectives and injectives and H be the associated half exact functor. The heart of $(\mathcal{U},\mathcal{V})$ has enough projectives $H(\Omega C)$ if and only if $(\mathcal{C},\mathcal{K})$ is a cotorsion pair on $\mathcal{E}$ where $\mathcal{K} = \mathrm{add}(\mathcal{U} * \mathcal{V})$ is the kernel of H. Moreover, when the heart of $(\mathcal{U},\mathcal{V})$ has enough projectives $H(\Omega\mathcal{C})$, it is equivalent to the functor category* $\mathrm{mod}(\mathcal{C}/\mathcal{P})$*, where $\mathcal{P}$ is the subcategory of projetive objects in $\mathcal{E}$.*

We also show that the condition $(\mathcal{C},\mathcal{K})$ is a torsion pair on triangulated category is satisfied in many cases, for example, when $\mathcal{U}$ is covariantly finite. And for exact category case, see Examples 4.3.

Key words and phrases. cotorsion pair, heart, enough projectives, functor category.

2. Hearts on triangulated categories

Let $\mathcal{T}$ be a triangulated category.

Definition 2.1. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{T}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{T}}(\mathcal{U},\mathcal{V})=0$.
(b) For any object $T\in\mathcal{T}$, there exists a triangle

$$T[-1]\to V_T\to U_T\to T$$

satisfying $U_T\in\mathcal{U}$ and $V^T\in\mathcal{V}$.

For a cotorsion pairs $(\mathcal{U},\mathcal{V})$, let $\mathcal{W}:=\mathcal{U}\cap\mathcal{V}$. We denote the quotient of $\mathcal{T}$ by $\mathcal{W}$ as $\underline{\mathcal{T}}:=\mathcal{T}/\mathcal{W}$. For any morphism $f\in\mathrm{Hom}_{\mathcal{T}}(X,Y)$, we denote its image in $\mathrm{Hom}_{\underline{\mathcal{T}}}(X,Y)$ by $\underline{f}$. For any subcategory $\mathcal{D}\supseteq\mathcal{W}$ of $\mathcal{T}$, we denote by $\underline{\mathcal{D}}$ the full subcategory of $\underline{\mathcal{T}}$ consisting of the same objects as $\mathcal{D}$. Let

$$\mathcal{T}^+:=\{T\in\mathcal{T}\mid U_T\in\mathcal{W}\},\quad \mathcal{T}^-:=\{T\in\mathcal{T}\mid V^T\in\mathcal{W}\}.$$

Let $\mathcal{H}:=\mathcal{T}^+\cap\mathcal{T}^-$ we call the additive subcategory $\underline{\mathcal{H}}$ the *heart* of cotorsion pair $(\mathcal{U},\mathcal{V})$. Under these settings, Abe, Nakaoka [AN] introduced the homological functor $H:\mathcal{T}\to\underline{\mathcal{H}}$ associated with $(\mathcal{U},\mathcal{V})$. Let $\mathcal{K}=\mathrm{add}(\mathcal{U}*\mathcal{V})$, as an ananlog of [L2, Proposition 4.7], we have $H(T)=0$ if and only if $T\in\mathcal{K}$.

In this section, let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair on $\mathcal{T}$.

For the coheart $\mathcal{C}:=\mathcal{U}[-1]\cap{}^{\perp}\mathcal{U}$, since $\mathcal{C}\subseteq\mathcal{T}^-$, for any object $C\in\mathcal{C}$, by definition of H we get the following commutative diagram the following commutative diagram

$$\begin{tikzcd}
V_C \arrow[r] \arrow[d, equal] & U_C \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "i"] & V_C[1] \arrow[d, equal] \\
V_C \arrow[r] & W_C \arrow[r] \arrow[d] & H(C) \arrow[r] \arrow[d] & V_C[1] \\
 & U'_C \arrow[r, equal] \arrow[d] & U'_C \arrow[d] & \\
 & U_C[1] \arrow[r] & C[1] &
\end{tikzcd} \tag{1}$$

where $U_C,U'_C\in\mathcal{U}$, $V_C\in\mathcal{V}$ and $W_C\in\mathcal{W}$. Moreover, $H(i)$ is an isomorphism in $\underline{\mathcal{H}}$ by [AN, Proposition 3.8, Theorem 5.7].

Definition 2.2. We denote by $H(\mathcal{C})$ the subcategory of $\underline{\mathcal{H}}$ such that every object $X\in H(\mathcal{C})$ admits a reflection triangle (see [AN, Definition 3.5] for details) $U[-1]\to C\xrightarrow{x}X\to U$ where $C\in\mathcal{C}$ and $U\in\mathcal{U}$.

Remark 2.3. By [AN, Remark 3.6], $H(x)$ is an isomorphism.

Let's start with an important property for H.

Proposition 2.4. *The functor $H|_{\mathcal{C}}:\mathcal{C}\to H(\mathcal{C})$ is an equivalence.*

Proof. By [AN, Remark 3.6] we get that H is dense on $\mathcal{C}$. We only have to check that $H|_{\mathcal{C}}$ is fully-faithful.
Let $C_1,C_2\in\mathcal{C}$, since C_i, $i=1,2$ admits a triangle $C_i\to H(C_i)\to U_i\to C_i[1]$ where $U_i\in\mathcal{U}$, let

$f \in \mathrm{Hom}_{\mathcal{T}}(C_1, C_2)$, by [N, Proposition 4.3], we get a commutative diagram

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d, "f"] & H(C_1) \arrow[r] \arrow[d, "f^+"] & U_1 \arrow[r] \arrow[d] & \mathcal{C}_1[1] \arrow[d] \\
C_2 \arrow[r] & H(C_2) \arrow[r] & U_2 \arrow[r] & C_2[1].
\end{tikzcd}$$

where $\underline{f}^+ = H(f)$. If $H(f) = 0$, f factors through $\mathcal{U}$ by [L2, Proposition 2.5]. Since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{U}) = 0$, we get $f = 0$ which means H is faithful on $\mathcal{C}$.

Let $g \in \mathrm{Hom}_{\mathcal{T}}(H(C_1), H(C_2))$, since $\mathrm{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{U}) = 0$, we can still get the following commutative diagram

$$\begin{tikzcd}
C_1 \arrow[r] \arrow[d, "f'"] & H(C_1) \arrow[r] \arrow[d, "g"] & U_1 \arrow[r] \arrow[d] & C_1[1] \arrow[d] \\
C_2 \arrow[r] & H(C_2) \arrow[r] & U_2 \arrow[r] & C_2[1]
\end{tikzcd}$$

Then we have $\underline{g} = H(f')$. Thus H is full on $\mathcal{C}$. $\square$

The following proposition is important for the subcategory $H(\mathcal{C})$ of $\underline{\mathcal{H}}$.

Proposition 2.5. *$H(\mathcal{C})$ is a subcategory of projectives in $\underline{\mathcal{H}}$. Moreover, it is closed under direct summands.*

Proof. We first prove that $H(\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.

Let $\underline{f} : A \to B$ be an epimorphism in $\underline{\mathcal{H}}$, since $A \in \mathcal{E}^-$, we get the following commutative diagram in $\mathcal{T}$

$$\begin{tikzcd}
U^A[-1] \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d, "f"] & W^A \arrow[r] \arrow[d] & U^A \arrow[d, equal] \\
U^A[-1] \arrow[r] & B \arrow[r, "g"'] & D \arrow[r] & U^A.
\end{tikzcd} \tag{2}$$

Since $\underline{f}$ is epimorphic, by [N2, Corollary 4.5], we get $D \in \mathcal{U}$.

Denote $B \oplus W^A$ by B', from the second square (2) we get a triangle $A \xrightarrow{f'} B' \xrightarrow{g'} D \to A[1]$ where $\underline{f'} = \underline{f}$.
Since $\mathrm{Hom}_{\mathcal{T}}(C, D) = 0$, $g'hx = 0$, we have the following commutative diagram

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, "j"] & X \arrow[r] \arrow[d, "h"] & U \arrow[r] \arrow[d] & C[1] \arrow[d] \\
A \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & D \arrow[r] & A[1].
\end{tikzcd}$$

Apply H to this diagram, since $H(x)$ is an isomorphism in $\underline{\mathcal{H}}$, we have the following commutative diagram

$$\begin{tikzcd}
& X \arrow[dl, "H(j)H(x)^{-1}"'] \arrow[d, "\underline{h}"] & \\
A \arrow[r, "\underline{f}"'] & B \arrow[r] & 0.
\end{tikzcd}$$

This implies that $H(\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.
Now we proof that $H(\mathcal{C})$ is closed under summands. Let $X_1 \oplus X_2 \in H(\mathcal{C})$, we get the following commutative diagram of triangles

$$\begin{tikzcd}
U[-1] \arrow[r] \arrow[d, equal] & T \arrow[r] \arrow[d] & X_1 \arrow[r, "u_1"] \arrow[d, "\binom{1}{0}"] & U \arrow[d, equal] \\
U[-1] \arrow[r, "b"] & C \arrow[r, "\binom{i_1}{i_2}"'] & X_1 \oplus X_2 \arrow[r, "(u_1\ u_2)"'] & U
\end{tikzcd}$$

It is easy to check that u_1 is a left U-approximation of X_1, hence we can get the following triangle $U'[-1] \xrightarrow{a} T' \longrightarrow X_1 \xrightarrow{u_1'} U'$ where u' is a minimal left U-approximation of X_1. Since $\operatorname{Hom}_{\mathcal{T}}(C, U') = 0$, we get the following commutative diagram of triangles

$$\begin{tikzcd}
U'[-1] \arrow[r, "a"] \arrow[d] & T' \arrow[r] \arrow[d, "f_1"] & X_1 \arrow[r, "u_1'"] \arrow[d, "\binom{1}{0}"] & U' \arrow[d, "g_1"] \\
U[-1] \arrow[r, "b"] \arrow[d] & C \arrow[r, "\binom{i_1}{i_2}"'] \arrow[d, "f_2"] & X_1 \oplus X_2 \arrow[r, "(u_1\ u_2)"'] \arrow[d, "(1\ 0)"] & U \arrow[d, "g_2"] \\
U'[-1] \arrow[r, "a"] & T' \arrow[r] & X_1 \arrow[r, "u_1'"] & U'
\end{tikzcd}$$

Since u' is minimal, we get that $g_2 g_1$ is an isomorphism, which implies that $f_2 f_1$ is also an isomorphism. Hence T' is a direct summand of C, which means that T' is also an object in $\mathcal{C}$. Now apply H to the above diagram, since $H(b) = 0$, we get $H(f_2 f_1)H(a) = 0$, which implies that $H(a) = 0$. Since $U'[-1]$ and T' lie in $\mathcal{B}^-$, by [L2, Proposition 2.5] a fators though $\mathcal{U}$. Hence by definition $U'[-1] \xrightarrow{a} T' \longrightarrow X_1 \xrightarrow{u_1'} U'$ is a reflection triangle and then $X_1 \in H(\mathcal{C})$. □

Proposition 2.6. *$\mathcal{C} = {}^{\perp}\mathcal{K}$.*

Proof. By definition $\mathcal{C} = \mathcal{U}[-1] \cap {}^{\perp}\mathcal{U} = {}^{\perp}\mathcal{V} \cap {}^{\perp}\mathcal{U}$, we get $\operatorname{Hom}_{\mathcal{T}}(\mathcal{C}, \mathcal{K}) = 0$ since $\mathcal{K} = \operatorname{add}(\mathcal{U} * \mathcal{V})$, this implies $\mathcal{C} \subseteq {}^{\perp}\mathcal{K}$. On the other hand, ${}^{\perp}\mathcal{K} \subseteq {}^{\perp}\mathcal{V} \cap {}^{\perp}\mathcal{U} = \mathcal{C}$. Hence $\mathcal{C} = {}^{\perp}\mathcal{K}$. □

Lemma 2.7. *The following conditions are equivalent to each other.*

(a) $\mathcal{H} \subseteq \mathcal{C} * \mathcal{K}$.
(b) $(\mathcal{C}, \mathcal{K})$ *is a torsion pair on* $\mathcal{T}$.
(c) $\mathcal{K}$ *is covariantly finite.*
(d) *Every object in* $\mathcal{H}$ *has a left* $\mathcal{K}$*-approximation.*

Proof. It is obvious that (b) implies (a). (a) and (d) (resp. (b) and (c)) are equivalent by Proposition 2.6 and Wakamatsu's Lemma. So it is enough to show (a) implies (b). We show it in two steps.
1. Let $T \in \mathcal{T}^-$, then it admits a reflection triangle $U[-1] \to T \xrightarrow{b} T^+ \to U$ where $T^+ \in \mathcal{H}$. Since $\mathcal{H} \subseteq \mathcal{C} * \mathcal{K}$, T^+ admits a triangle $C \xrightarrow{c} T^+ \to K \to C[1]$. $\operatorname{Hom}_{\mathcal{T}}(C, U) = 0$ by definition, so there exists a

morphism $d : C \to T$ such that $bd = c$. We can get the following commutative diagram.

$$\begin{tikzcd}
C \arrow[r, "d"] \arrow[d, equal] & T \arrow[r] \arrow[d, "b"] & D \arrow[r] \arrow[d] & C[1] \arrow[d, equal] \\
C \arrow[r, "c"] & T^{+} \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & C[1] \\
 & U \arrow[r, equal] \arrow[d] & U \arrow[d] & \\
 & B[1] \arrow[r] & D[1] &
\end{tikzcd}$$

Since $H(c)$ is an epimorphism and $H(b)$ is an isomorphism, $H(d)$ is also an epimorphism. Thus $H(D) = 0$ since $H(C[1]) = 0$. This implies that $\mathcal{T}^- \subseteq \mathcal{C} * \mathcal{K}$.
2. Let T be any object in $\mathcal{T}$, it admits a coreflection triangle $V \to T^- \to B \to V[1]$, since $\mathcal{T}^- \subseteq \mathcal{C} * \mathcal{K}$, T^- admits a tirangle $C \xrightarrow{x} T^- \to K \to C[1]$, hence we get the following commutative diagram by the octahedral axiom

$$\begin{tikzcd}
 & V \arrow[r, equal] \arrow[d] & V \arrow[d] & \\
C \arrow[r, "x"] \arrow[d, equal] & T^{-} \arrow[r] \arrow[d, "f"] & K \arrow[r] \arrow[d] & C[1] \arrow[d, equal] \\
C \arrow[r, "fx"] & T \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & C[1] \\
 & V[1] \arrow[r, equal] & V[1]. &
\end{tikzcd}$$

Since $H(x)$ is an isomorphism and $\underline{f}$ is an epimorphism, we get $H(fx)$ is an epimorphism. H(C[1])=0 since $C[1] \in \mathcal{U}$, hence $H(D) = 0$, which means that $D \in \mathcal{K}$. Then $T \in \mathcal{C} * \mathcal{K}$. Hence $\mathcal{T} = \mathcal{C} * \mathcal{K}$ and $(\mathcal{C}, \mathcal{K})$ is a torsion pair by Proposition 2.6. □

Now we prove the following theorems.

Theorem 2.8. *$\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$ if and only if $(\mathcal{C}, \mathcal{K})$ is a torsion pair.*

Proof. We show the if part first.
If $(\mathcal{C}, \mathcal{K})$ is a torsion pair, any object $A \in \mathcal{H}$ admits a triangle $C_A \to A \to K_A \to C_A[1]$ where $C_A \in \mathcal{C}$ and $K_A \in K$, apply H to this triangle, we get an exact sequence $H(C_A) \to A \to 0$ in $\underline{\mathcal{H}}$, which means $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$.
Now we show the only if part.
Let $T \in \mathcal{H}$, if $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$, there exists a morphism $\underline{f} : X \to B$ where $X \in H(\mathcal{C})$ and admitting a triangle $C \xrightarrow{x} X \to U \to C[1]$ where $C \in \mathcal{C}$ and $U \in \mathcal{U}$. We have the following commutative

diagram by the octahedral axiom.

$$\begin{tikzcd}
C \arrow[r, "x"] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & U \arrow[r] \arrow[d] & C[1] \arrow[d, equal] \\
C \arrow[r, "fx"] & B \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & C[1] \\
& Y \arrow[r, equal] \arrow[d] & Y \arrow[d] & \\
& X[1] \arrow[r] & U[1] &
\end{tikzcd}$$

Since $H(x)$ is an isomorphism and $\underline{f}$ is an epimorphism, we get $H(fx)$ is an epimorphism. H(C[1])=0 since $C[1] \in \mathcal{U}$, hence $H(D) = 0$, which means $D \in \mathcal{K}$. Then we get $\mathcal{H} \subseteq \mathcal{C} * \mathcal{K}$. Now by Proposition 2.7, we have $(\mathcal{C}, \mathcal{K})$ is a torsion pair. $\square$

We also have the following corollary as an observation.

Corollary 2.9. *If $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$, then every projective object of $\underline{\mathcal{H}}$ lies in $H(\mathcal{C})$.*

Proof. By Theorem 2.8, if $\underline{\mathcal{H}}$ has enough projectives $H(\mathcal{C})$, then $\mathcal{T} = \mathcal{C} * \mathcal{K}$. Let $P \in \underline{\mathcal{H}}$ be a projective obejct, it admits a triangle $C_P \to P \to K_P \to C_P[1]$, apply H to this triangle, we get an exact sequence $H(C_P) \to P \to 0$ in $\underline{\mathcal{H}}$. Hence P is a direct summand of $H(C_P)$. By Proposition 2.5, P lies in $H(\mathcal{C})$. $\square$

Theorem 2.10. *If $(\mathcal{C}, \mathcal{K})$ is a torsion pair, then $\underline{\mathcal{H}} \simeq \operatorname{mod} \mathcal{C}$.*

Proof. It is enough to show that $\underline{\mathcal{H}} \simeq \operatorname{mod} H(\mathcal{C})$ since $\mathcal{C} \simeq H(\mathcal{C})$.
Define

$$\begin{aligned} F : \mathcal{H} &\to \operatorname{mod} H(\mathcal{C}) \\ A &\mapsto \operatorname{Hom}_{\underline{\mathcal{T}}}(-, A)|_{H(\mathcal{C})}. \end{aligned}$$

Now we show that F is dense.
Let $N \in \operatorname{mod} H(\mathcal{C})$, we have an exact sequence

$$\operatorname{Hom}_{H(\mathcal{C})}(-, P_1) \xrightarrow{\operatorname{Hom}_{H(\mathcal{C})}(-,\underline{f})} \operatorname{Hom}_{H(\mathcal{C})}(-, P_0) \to N \to 0$$

where $P_1, P_0 \in H(\mathcal{C})$. Since $\underline{\mathcal{H}}$ is abelian, we have a exact sequence $P_1 \xrightarrow{\underline{f}} P_0 \to Y \to 0$ Now apply $\operatorname{Hom}_{\underline{\mathcal{T}}}(H(\mathcal{C}), -)$ to this exact sequence, we have

$$\operatorname{Hom}_{H(\mathcal{C})}(-, P_1) \xrightarrow{\operatorname{Hom}_{H(\mathcal{C})}(-,\underline{f})} \operatorname{Hom}_{H(\mathcal{C})}(-, P_0) \to \operatorname{Hom}_{\underline{\mathcal{T}}}(-, H(Y))|_{H(\mathcal{C})} \to 0.$$

Hence $N \simeq \operatorname{Hom}_{\underline{\mathcal{T}}}(-, H(Y))|_{H(\mathcal{C})}$.
We prove that F is faithful.
Let $\underline{f} : A \to B$ be a morphism in $\underline{\mathcal{H}}$ such that $F(\underline{f}) = 0$. Since $\mathcal{T} = \mathcal{C} * \mathcal{K}$, A admits a triangle $C_A \xrightarrow{i} A \to K \to C_A[1]$, and C_A admits a triangle $C_A \xrightarrow{g} H(C) \xrightarrow{h} U \to C[1]$. Since there exists a morphism $j : H(C) \to A$ such that $i = jg$, we have $\underline{fj} = 0$, hence $fi = fjg$ factors through $\mathcal{W}$, then $\underline{f}H(i) = 0$. Since $H(i)$ is epimorphic, we get $\underline{f} = 0$.
We prove that F is full.
Let $\alpha : \operatorname{Hom}_{\underline{\mathcal{T}}}(-, A_1)|_{H(\mathcal{C})} \to \operatorname{Hom}_{\underline{\mathcal{T}}}(-, A_2)|_{H(\mathcal{C})}$ be a morphism in $\operatorname{mod} H(\mathcal{C})$. By Theorem 2.8, A_i admits an exact sequence $P'_{A_i} \xrightarrow{\underline{g_i}} P_{A_i} \xrightarrow{\underline{f_i}} A_i \to 0$ such that $P'_{A_i}, P_{A_i} \in H(\mathcal{C})$, we get the following

commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{H(\mathcal{C})}(-,P'_{A_1}) \arrow[r, "\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{g_1})"] \arrow[d, "\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{a})"] & \mathrm{Hom}_{H(\mathcal{C})}(-,P_{A_1}) \arrow[r, "\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{f_1})"] \arrow[d, "\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{b})"] & \mathrm{Hom}_{\underline{\mathcal{T}}}(-,A_1)|_{H(\mathcal{C})} \arrow[r] \arrow[d, "\alpha"] & 0 \\
\mathrm{Hom}_{H(\mathcal{C})}(-,P'_{A_2}) \arrow[r, "\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{g_2})"'] & \mathrm{Hom}_{H(\mathcal{C})}(-,P_{A_2}) \arrow[r, "\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{f_2})"'] & \mathrm{Hom}_{\underline{\mathcal{T}}}(-,A_2)|_{H(\mathcal{C})} \arrow[r] & 0
\end{tikzcd}$$

by Yoneda's Lemma. Hence we get the following commutative commutative diagram

$$\begin{tikzcd}
P'_{A_1} \arrow[r, "\underline{g_1}"] \arrow[d, "\underline{a}"] & P_{A_1} \arrow[r, "\underline{f_1}"] \arrow[d, "\underline{b}"] & A_1 \arrow[r] \arrow[d, dotted, "\underline{c}"] & 0 \\
P'_{A_2} \arrow[r, "\underline{g_2}"] & P_{A_2} \arrow[r, "\underline{f_2}"] & A_2 \arrow[r] & 0.
\end{tikzcd}$$

Hence $\mathrm{Hom}_{H(\mathcal{C})}(-,\underline{c}) = \alpha$. □

Note that the condition $(\mathcal{C},\mathcal{K})$ is a torsion pair is satisfied in many cases. We give the following proposition as an exmaple.

Proposition 2.11. *If $\mathcal{U}$ is covariantly finite, then $(\mathcal{C},\mathcal{K})$ is a torsion pair.*

Proof. If $\mathcal{U}$ is covariantly finite, then $({}^{\perp_1}\mathcal{U},\mathcal{U})$ is a cotorsion pair. Hence any object $U \in \mathcal{U}$ admits a triangle $U' \to C_U[1] \to U \to U'[1]$ where $C_U[1] \in \mathcal{U} \cap {}^{\perp_1}\mathcal{U} = \mathcal{C}[1]$, which implies that $\mathcal{U}[-1] \subseteq \mathcal{C} * \mathcal{U}$. Let T be any objects in $\mathcal{T}$, it admits a triangle $U_T[-1] \to T \to V_T \to U_T$, since $U_T[-1]$ admits a triangle $C \to U_T[-1] \to U \to C[1]$, we get the following commutative diagram by the octahedral axiom

$$\begin{tikzcd}
C \arrow[r] \arrow[d, equal] & U_T[-1] \arrow[r] \arrow[d] & U \arrow[r] \arrow[d] & C[1] \arrow[d, equal] \\
C \arrow[r] & T \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & C[1] \\
 & V_T \arrow[r, equal] \arrow[d] & V_T \arrow[d] & \\
 & U_T \arrow[r] & U[1] &
\end{tikzcd}$$

where $K \in \mathcal{U} * \mathcal{V} \subseteq \mathcal{K}$. Hence $\mathcal{T} = \mathcal{C} * \mathcal{K}$ and then $(\mathcal{C},\mathcal{K})$ is a torsion pair. □

One special case for the condition $\mathcal{T} = \mathcal{C} * \mathcal{K}$ is that $\mathcal{U}$ is rigid. In this case, we have $\mathcal{C} = \mathcal{U}[-1]$ and $\mathcal{K} = \mathcal{V}$. This case has been discussed in [N, Section 7], see [N, Corollary 7.4] for details. We have the following corollary.

Corollary 2.12. *Let $(\mathcal{U},\mathcal{V})$ be a cotorsion pair where $\mathcal{U}$ is rigid, then $\underline{\mathcal{H}}$ has enough projectivs $H(\mathcal{U}[-1])$, and it is equivalent to* $\mathrm{mod}(\mathcal{U}[-1])$.

Let $\mathcal{V}^{\perp} = \{T \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(\mathcal{V},T) = 0\}$ and $\mathcal{D} = \mathcal{V}[1] \cap \mathcal{V}^{\perp}$. At the end of this section, we introduce the following theorem which is the dual of Theorem 2.8.

Theorem 2.13. *$\underline{\mathcal{H}}$ has enough injectives $H(\mathcal{D})$ if and only if $(\mathcal{K},\mathcal{D})$ is a torsion pair.*

3. Hearts on exact categories

Let $\mathcal{E}$ be a Krull-Schmidt, Hom-finite, k-linear exact category with enough projectives and enough injectives. Let $\mathcal{P}$ (resp. $\mathcal{I}$) be the subcategory of projective (resp. injective) objects.

Definition 3.1. Let $\mathcal{U}$ and $\mathcal{V}$ be full additive subcategories of $\mathcal{E}$ which are closed under direct summands. We call $(\mathcal{U},\mathcal{V})$ a *cotorsion pair* if it satisfies the following conditions:

(a) $\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{U},\mathcal{V})=0$.
(b) For any object $B\in\mathcal{E}$, there exits two short exact sequences

$$0\to V_B\to U_B\to B\to 0,\quad 0\to B\to V^B\to U^B\to 0$$

satisfying $U_B,U^B\in\mathcal{U}$ and $V_B,V^B\in\mathcal{V}$.

For a cotorsion pairs $(\mathcal{U},\mathcal{V})$, we denote the quotient of $\mathcal{E}$ by $\mathcal{U}\cap\mathcal{V}$ as $\underline{\mathcal{E}}:=\mathcal{E}/\mathcal{U}\cap\mathcal{V}$. Denote $\mathcal{U}\cap\mathcal{V}$ by $\mathcal{W}$, for any morphism $f\in\mathrm{Hom}_{\mathcal{E}}(X,Y)$, we denote its image in $\mathrm{Hom}_{\underline{\mathcal{E}}}(X,Y)$ by $\underline{f}$. For any subcategory $\mathcal{D}\supseteq\mathcal{W}$ of $\mathcal{T}$, we denote by $\underline{\mathcal{D}}$ the full subcategory of $\underline{\mathcal{E}}$ consisting of the same objects as $\mathcal{D}$. Let

$$\mathcal{E}^+:=\{B\in\mathcal{E}\mid U_B\in\mathcal{W}\},\quad \mathcal{E}^-:=\{B\in\mathcal{E}\mid V^B\in\mathcal{W}\}.$$

Let $\mathcal{H}:=\mathcal{E}^+\cap\mathcal{E}^-$, we denote the additive subcategory $\underline{\mathcal{H}}$ the *heart* of cotorsion pair $(\mathcal{U},\mathcal{V})$. Let $H:\mathcal{E}\to\underline{\mathcal{H}}$ be the half exact functor associated with $(\mathcal{U},\mathcal{V})$.

Let $\Omega\mathcal{C}=\{X\in\mathcal{E}\mid \text{X admits } 0\to X\to P\to C\to 0 \text{ where } P\in\mathcal{P} \text{ and } C\in\mathcal{C}\}$, since $\Omega\mathcal{C}\subseteq\mathcal{E}^-$ by [L2, Lemma 3.2], for any object $\Omega C\in\Omega\mathcal{C}$, by definition of H we get from the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & V_{\Omega C} \arrow[r] \arrow[d, equal] & U_{\Omega C} \arrow[r] \arrow[d] & \Omega C \arrow[r] \arrow[d, "a"] & 0 \\
0 \arrow[r] & V_{\Omega C} \arrow[r] & W_{\Omega C} \arrow[r] \arrow[d] & H(\Omega C) \arrow[r] \arrow[d] & 0 \\
 & & U \arrow[r, equal] \arrow[d] & U \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where $H(a)$ is an isomorphism by [L2, Theorem 4.1, Proposition 4.2].

Definition 3.2. We denote by $H(\Omega\mathcal{C})$ the subcategory of $\underline{\mathcal{H}}$ such that every object $X\in H(\Omega\mathcal{C})$ admits a reflection sequence (see [L2, Definition 3.3] for details) $0\to\Omega C\xrightarrow{x}X\to U\to 0$ where $C\in\mathcal{C}$ and $U\in\mathcal{U}$.

Since ΩC admits a short exact sequence $0 \to \Omega C \to P \to C \to 0$ where $P \in \mathcal{P}$ and $C \in \mathcal{C}$, we have the following commuative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Omega C \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "u"] & U \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P \arrow[r] \arrow[d] & U_X \arrow[r] \arrow[d] & U \arrow[r] & 0 \\
 & C \arrow[r, equal] \arrow[d] & C \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

where the second row and second column are short exact sequences. We have $U_X \in \mathcal{U}$ since $\mathcal{U}$ is closed under extension and $\mathcal{P} \subseteq \mathcal{U}$. According to this diagram, we have the following lemma.

Lemma 3.3. *Every object $X \in H(\Omega\mathcal{C})$ admits a short exact seqeunce $0 \to X \to U_X \to C \to 0$ where $U_X \in \mathcal{U}$ and $C \in \mathcal{C}$.*

Remark 3.4. By [L2, Theorem 4.1,Proposition 4.7], $H(x)$ is an isomorphism in $\underline{\mathcal{H}}$.

We will prove the following theorem.

Theorem 3.5. *Let $(\mathcal{U}, \mathcal{V})$ be a cotorsion pair. Let $\mathcal{C} := \mathcal{U} \cap {}^{\perp_1}\mathcal{U}$ and $\Omega\mathcal{C} = \{X \in \mathcal{E} \mid X$ admits $0 \to X \to P \to C \to 0$ where $P \in \mathcal{P}$ and $C \in \mathcal{C}\}$. Then the heart of $(\mathcal{U}, \mathcal{V})$ has enough projectives $H(\Omega\mathcal{C})$ if and only if $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair. Moreover, when the heart of $(\mathcal{U}, \mathcal{V})$ has enough projectives $H(\Omega\mathcal{C})$, it is equivalent to* $\operatorname{mod}(\mathcal{C}/\mathcal{P})$.

We prove the theorem in several steps. We denote the quotient of $\mathcal{E}$ by $\mathcal{P}$ as $\overline{\mathcal{E}} := \mathcal{E}/\mathcal{P}$. For any morphism $f \in \operatorname{Hom}_{\mathcal{E}}(X, Y)$, we denote its image in $\operatorname{Hom}_{\overline{\mathcal{E}}}(X, Y)$ by $\overline{f}$.

Lemma 3.6. $\overline{\mathcal{C}} \simeq \overline{\Omega\mathcal{C}}$.

Proof. For any morphism $f : C \to C'$ in $\mathcal{C}$, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d, "g"] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega C' \arrow[r] & P' \arrow[r] & C' \arrow[r] & 0.
\end{tikzcd}$$

We can define a functor $G : \overline{\mathcal{C}} \to \overline{\Omega\mathcal{C}}$ such that $G(C) = \Omega C$ and $G(\overline{f}) = \overline{g}$. G is well defined since if f factors through $P'' \in \mathcal{P}$, then it factors through P', which implies g factors through P, hence $\overline{g} = 0$. We prove that G is an equivalence.
(i) We first prove that G is faithful.
If $\overline{g} = 0$, it factors through an projective object P_0. By the definition of $\mathcal{C}$, we get $\operatorname{Ext}^1_{\mathcal{E}}(\mathcal{C}, \mathcal{P}) = 0$, hence we have the following

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[rr] \arrow[dd, "g"] \arrow[dr] & & P \arrow[r] \arrow[dd] \arrow[dl, dotted] & C \arrow[r] \arrow[dd, "f"] & 0 \\
 & & P_0 \arrow[dl] & & & \\
0 \arrow[r] & \Omega C' \arrow[rr] & & P' \arrow[r] & C' \arrow[r] & 0.
\end{tikzcd}$$

This implies that f factors through P', hence $\overline{f}=0$.
(ii) We prove that G is full.
For the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d, "g"] & P \arrow[r] & C \arrow[r] & 0 \\
0 \arrow[r] & \Omega C' \arrow[r] & P' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

since $\mathrm{Ext}^1_{\mathcal{E}}(\mathcal{C},\mathcal{P})=0$, we can get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d, "g"] & P \arrow[r] \arrow[d, dotted] & C \arrow[r] \arrow[d, dotted, "f"] & 0 \\
0 \arrow[r] & \Omega C' \arrow[r] & P' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

hence $G(\overline{f})=\overline{g}$.
By the definition of $\Omega\mathcal{C}$, G is dense. Hence G is an equivalence. □

Since $H(\mathcal{P})=0$, we have the following commutative diagram

$$\begin{tikzcd}
\Omega\mathcal{C} \arrow[rr, "H"] \arrow[dr, "\pi"'] & & H(\Omega\mathcal{C}) \\
& \overline{\Omega\mathcal{C}} \arrow[ur, "\overline{H}"'] &
\end{tikzcd}$$

where π is the quotient functor.

Proposition 3.7. $\overline{H}:\overline{\Omega\mathcal{C}}\to H(\Omega\mathcal{C})$ *is an equivalence.*

Proof. By [L2, Lemma 3.3] we get $\overline{H}$ is dense. Now we only have to check that $\overline{H}$ is fully-faithful.
Let $\Omega C_i\in\Omega\mathcal{C}$, $i=1,2$, it admits a short exact sequence

$$0\to\Omega C_i\to H(\Omega C_i)\to U_i\to 0$$

where $U_i\in\mathcal{U}$. Let $\overline{f}\in\mathrm{Hom}_{\overline{\mathcal{E}}}(\Omega C_1,\Omega C_2)$, by [L, Proposition 3.3], we get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C_1 \arrow[r] \arrow[d, "f"] & H(\Omega C_1) \arrow[r] \arrow[d, "f^+"] & U_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega C_2 \arrow[r] & H(\Omega C_2) \arrow[r] & U_2 \arrow[r] & 0
\end{tikzcd}$$

where $\underline{f}^+=H(f)$. If $H(f)=0$, f factors through $\mathcal{U}$ by [L2, Proposition 2.5]. Since $\mathrm{Hom}_{\overline{\mathcal{E}}}(\Omega\mathcal{C},\mathcal{U})=0$, we get $\overline{f}=0$ which means $\overline{H}$ is faithful on $\overline{\Omega\mathcal{C}}$.
Let $\underline{g}\in\mathrm{Hom}_{\underline{\mathcal{E}}}(H(\Omega C_1),H(\Omega C_2))$, since $\mathrm{Hom}_{\overline{\mathcal{E}}}(\Omega\mathcal{C},\mathcal{U})=0$, we get that in the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C_1 \arrow[r, "a_1"] & H(\Omega C_1) \arrow[r, "b_1"] \arrow[d, "g"] & U_1 \arrow[r] & 0 \\
0 \arrow[r] & \Omega C_2 \arrow[r, "a_2"'] & H(\Omega C_2) \arrow[r, "b_2"'] & U_2 \arrow[r] & 0.
\end{tikzcd}$$

b_2ga_1 factors through an object $P\in\mathcal{P}$. Hence we have two morphisms $c:\Omega C_1\to P$ and $d:P\to U_2$ such that $dc=b_2ga_1$. Since P is projective, there exists a morphism $p:P\to H(\Omega C_2)$ such that $d=b_2p$. Hence $b_2(ga_1-pc)=0$. Then there is a morphism $f':\Omega C_1\to\Omega C_2$ such that $f'a_2=ga_1-pc$. now we

get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C_1 \arrow[r, "a_1"] \arrow[d, "f'"] & H(\Omega C_1) \arrow[r, "b_1"] \arrow[d, "g'"] & U_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Omega C_2 \arrow[r, "a_2"'] & H(\Omega C_2) \arrow[r, "b_2"'] & U_2 \arrow[r] & 0.
\end{tikzcd}$$

by [L, Proposition 3.3] where $H(f') = \underline{g'}$. Since $H(\mathcal{P}) = 0$, we get $\underline{g'}H(a_1) = H(f')H(a_2) = \underline{g}H(a_1)$. Since $H(a_1)$ is an isomorphism, we have $\underline{g'} = \underline{g}$. Hence $\overline{H}$ is full. $\square$

The following lemma which can be regarded as an exact category version of Wakamatsu's Lemma will be needed.

Lemma 3.8. *Let $\mathcal{D}$ be an extension-closed subcategory of $\mathcal{E}$ which contains $\mathcal{I}$. If an object $A \in \mathcal{E}$ has a left $\mathcal{D}$-approximation, then A admits a short exact sequence $0 \to A \xrightarrow{f} D \to B$ where f is a minimal left $\mathcal{D}$-approximation and $B \in {}^{\perp_1}\mathcal{D}$.*

The following proposition is an analog of Proposition 2.6.

Proposition 3.9. $\mathcal{C} = {}^{\perp_1}\mathcal{K}$.

Proposition 3.10. *$H(\Omega\mathcal{C})$ is a subcategory of projectives in $\underline{\mathcal{H}}$. Moreover, it is closed under direct summands.*

Proof. We first prove that $H(\Omega\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.
Let $\underline{f} : A \to B$ be an epimorphism in $\underline{\mathcal{H}}$, it admits the following commutative diagram in $\mathcal{E}$

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & W^A \arrow[r] \arrow[d] & U^A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "g"'] & D \arrow[r] & U^A \arrow[r] & 0.
\end{tikzcd}$$

By [L, Corollary 3.11], we have $D \in \mathcal{U}$. Now we can assume that f admits a short exact sequence: $0 \to A \xrightarrow{f'} B' \xrightarrow{g'} D \to 0$ such that $D \in U$, $\underline{f} = \underline{f'}$ and $B = B'$ in $\underline{\mathcal{H}}$. Now let $X \in H(\Omega\mathcal{C})$ and $\underline{h} : X \to B$ be a morphism in $\underline{\mathcal{H}}$, we get the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r, "x"] & X \arrow[r] \arrow[d, "h"] & U \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"] & B' \arrow[r, "g'"] & D \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{Hom}_{\overline{\mathcal{E}}}(\Omega C, U) = 0$, $g'hx$ factors through $\mathcal{P}$. Hence as in the proof of Proposition 3.7, we can get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r, "x"] \arrow[d, "j"] & X \arrow[r] \arrow[d, "h'"] & U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "f'"] & B' \arrow[r, "g'"] & D \arrow[r] & 0
\end{tikzcd}$$

where $\underline{h} = \underline{h'}$. Since $H(x)$ is an isomorphism in $\underline{\mathcal{H}}$, we have the following commutative diagram

$$\begin{tikzcd}
 & X \arrow[dl, "H(j)H(x)^{-1}"'] \arrow[d, "\underline{h}"] & \\
A \arrow[r, "\underline{f}"'] & B \arrow[r] & 0.
\end{tikzcd}$$

This implies that $H(\Omega\mathcal{C})$ is projective in $\underline{\mathcal{H}}$.
Now we show that $H(\Omega\mathcal{C})$ is closed under direct summands.
Let $X_1\oplus X_2\in H(\Omega\mathcal{C})$, By Lemma 3.3, it admits a short exact sequence $0\to X_1\oplus X_2\xrightarrow{(u_1\ u_2)} U\to C$. Since $C\in {}^{\perp_1}\mathcal{K}$, we get $X_1\oplus X_2\xrightarrow{(u_1\ u_2)} U$ is a left $\mathcal{K}$-approximation, hence $X_1\xrightarrow{u_1} U$ is also a left $\mathcal{K}$-approximation. By Lemma 3.8, X_1 admits a short exact sequence $0\to X_1\xrightarrow{k_1} K_1\to C_1\to 0$ where k_1 is a minimal left $\mathcal{K}$-approximation, $K_1\in\mathcal{K}$ and $C_1\in\mathcal{C}$. We get thw following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r, "k_1"] \arrow[d, "\binom{1}{0}"'] & K_1 \arrow[r] \arrow[d, "f"] & C_1 \arrow[r] \arrow[d, "a"] & 0\\
0 \arrow[r] & X_1\oplus X_2 \arrow[r, "(u_1\ u_2)"] \arrow[d, "(1\ 0)"'] & U \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "b"] & 0\\
0 \arrow[r] & X_1 \arrow[r, "k_1"] & K_1 \arrow[r] & C_1 \arrow[r] & 0.
\end{tikzcd}$$

Since k_1 is minimal, we have gf is an isomorphism, hence ba is also an isomorphism. Since $\mathcal{U}$ is closed under direct summands, we get $K_1\in\mathcal{U}$. By [L2, Corollary 4.6], we get the following commutative diagram in $\underline{\mathcal{H}}$.

$$\begin{tikzcd}
H(\Omega K_1) \arrow[r, "\alpha"] \arrow[d] & H(\Omega C_1) \arrow[r] \arrow[d, "H(\Omega a)"] & H(X_1) \arrow[d, "\binom{1}{0}"]\\
H(\Omega U) \arrow[r, "\beta"] \arrow[d] & H(\Omega C) \arrow[r] \arrow[d, "H(\Omega b)"] & H(X_1)\oplus H(X_2) \arrow[d, "(1\ 0)"]\\
H(\Omega K_1) \arrow[r] & H(\Omega C_1) \arrow[r] & H(X_1)
\end{tikzcd}$$

since ba is isomorphic, $\overline{\Omega a\Omega b}$ is also isomorphic by Lemma 3.6. Hence $H(\Omega b)H(\Omega a)$ is an isomorphism. β is zero by the definition of $H(\Omega\mathcal{C})$, hence α is also zero and we get a reflection sequence $0\to\Omega C_1\to X_1\oplus P\to K_1\to 0$ where $P\in\mathcal{P}\cap\mathcal{W}$, which implies $X_1\in H(\Omega\mathcal{C})$. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 3.11. *$\underline{\mathcal{H}}$ has enough projectives $H(\Omega\mathcal{C})$ if and only if $(\mathcal{C},\mathcal{K})$ is a cotorsion pair on $\mathcal{E}$.*

Proof. We proof the if part first.
Since $(\mathcal{C},\mathcal{K})$ is a cotorsion pair, any object $B\in\mathcal{H}$ admits a short exact sequence $0\to B\to K\to C\to 0$ where $K\in\mathcal{K}$ and $C\in\mathcal{C}$. Hence we get the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & B \arrow[r] & K \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

which implies that $H(f):H(\Omega C)\to B$ is an epimorphism. Hence $\underline{\mathcal{H}}$ has enough projectives $H(\Omega\mathcal{C})$.
Now we prove the only if part. By Proposition 3.9 and the dual of [L, Proposition 2.12], it is enough to show that $\mathcal{K}$ is covariantly finite. We prove it in three steps.
1. Let $B\in\mathcal{H}$, then it admits a epimorphism $X\xrightarrow{x} B$ in $\underline{\mathcal{H}}$ where $X\in H(\Omega C)$. Since X amdits a short exact sequence $0\to X\to U_X\to C$ where $U_X\in\mathcal{U}$ and $C\in\mathcal{C}$, we get the following commutative diagram by using push-out.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "x"] & U_X \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0\\
0 \arrow[r] & B \arrow[r, "k"'] & K \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

Since $H(k)\underline{x} = 0$ and $\underline{x}$ is epimophic, we get $H(k) = 0$. Thus $H(K) = 0$ since $H(C) = 0$, which means $K \in \mathcal{K}$. Then k is a left $\mathcal{K}$-approximation since $C \in \mathcal{C} = {}^{\perp_1}\mathcal{K}$.

2. Let $B \in \mathcal{E}^+$, then B admits a short exact sequence $0 \to V \to B^- \xrightarrow{b^-} B \to 0$ where $B^- \in \mathcal{H}$. By step 1 it admits a short exact sequence $0 \to B^- \to K \to C \to 0$, hence we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B^- \arrow[r] \arrow[d, "b^-"] & K \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "k'"] & K' \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

Since $H(b^-)$ is an isomorphism, as in step 1, k' is a left $\mathcal{K}$-approximation.

3. Let $B \in \mathcal{E}$, B admits a short exact squence $0 \to B \xrightarrow{b} B^+ \xrightarrow{u} U \to 0$ where $B^+ \in \mathcal{E}^+$. By step 2, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r, "q"] \arrow[d, "a"] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B^+ \arrow[r] & K \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

where $H(a)$ is an epimorphism. Since $\mathrm{Ext}^1_{\mathcal{E}}(C, U) = 0$, there exists a morphism $p_U : P \to U$ such that $ua = p_U q$. Since P is projective, there is a morphism $p : P \to B^+$ such that $p_U = up$, then $ua - p_U q = ua - upq = u(a - pq) = 0$. Hence there is a morphism $d : \Omega C \to B$ such that $a - pq = b^+d$. Since $H(b^+)$ is an isomorphism, $H(d)$ is also an epimorphism. By using push-out, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega C \arrow[r] \arrow[d, "d"] & P \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "k'"] & K' \arrow[r] & C \arrow[r] & 0.
\end{tikzcd}$$

As in step 1, k' is a left $\mathcal{K}$-approximation. Hence $\mathcal{K}$ is covariantly finite, and then $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair on $\mathcal{E}$. □

From the proof of this theorem we get the folloiwng corollary, which is an analog of Lemma 2.7.

Corollary 3.12. *The following conditions are equivalent to each other.*

(a) *Every object in $\mathcal{H}$ admits a left $\mathcal{K}$-approximation.*
(b) *$(\mathcal{C}, \mathcal{K})$ is a cotorsion pair.*

The following corollary is an analog of Corollary 2.9

Corollary 3.13. *If $\underline{\mathcal{H}}$ has enough projectives $H(\Omega\mathcal{C})$, then every projective object of $\underline{\mathcal{H}}$ lies in $H(\Omega\mathcal{C})$.*

Theorem 3.14. *If $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair, then $\underline{\mathcal{H}} \simeq \mathrm{mod}\,\overline{\mathcal{C}}$.*

Proof. This is an analog of Theorem 2.10. □

Note that the condition $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair is satisfied in many cases. We give the following proposition as an example.

Proposition 3.15. *If $\mathcal{U}$ is covariantly finite and contains $\mathcal{I}$, then $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair.*

Proof. This is an analog of Proposition 2.11. □

One special case of the condition $(\mathcal{C}, \mathcal{K})$ is a cotorsion pair is that $\mathcal{U}$ is rigid. In this case, $\mathcal{C} = \mathcal{U}$ and $\mathcal{K} = \mathcal{V}$. This case has been discussed in [DL], for details ,see [DL, Theorem 3.2]

Let $\mathcal{V}^{\perp_1} = \{X \in \mathcal{E} \mid \mathrm{Ext}^1_{\mathcal{E}}(\mathcal{V}, X) = 0\}$ and $\mathcal{D} = \mathcal{V} \cap \mathcal{V}^{\perp_1}$, let $\Omega^-\mathcal{D} = \{X \in \mathcal{E} \mid \text{X admits } 0 \to D \to I \to X \to 0 \text{ where } I \in \mathcal{I} \text{ and } D \in \mathcal{D}\}$. At the end of this section, we introduce the following theorem which is the dual of Theorem 3.11.

Theorem 3.16. *$\underline{\mathcal{H}}$ has enough injectives $H(\Omega^-\mathcal{D})$ if and only if $(\mathcal{K}, \mathcal{D})$ is a cotorsion pair.*

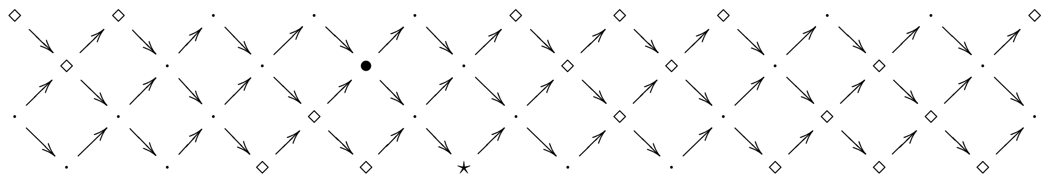
4. Examples

In this section we give several examples of our main theorem.

The first example comes from [KZ, Corollary 4.4].

Example 4.1. Let $\mathcal{M}$ be a cluster tilting subcategory of $\mathcal{T}$, then $(\mathcal{M},\mathcal{M})$ is a cotorsion pair with coheart $\mathcal{M}[-1]$. This cotorsion pair satisfies the condition in Theorem 2.10, we get an equivalence $\mathcal{T}/\mathcal{M} \simeq \operatorname{mod}(\mathcal{M}[-1])$ where $\mathcal{T}/\mathcal{M}$ is the heart of $(\mathcal{M},\mathcal{M})$.

Example 4.2. Let k be a field.



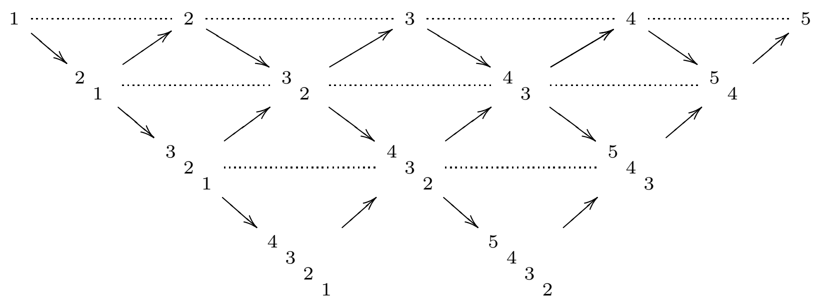
The above diagram is a part of $D^b(\operatorname{mod} k\mathbf{A}_4)$ which continues infinitely in both sides. Let $\mathcal{U}$ be the objects in $\diamond$, then $(\mathcal{U},\mathcal{U}^{\perp_1})$ is a cotorsion pair. The coheart $\mathcal{C}$ of it is in $\bullet$, and the heart $\underline{\mathcal{H}}$ of $(\mathcal{U},\mathcal{U}^{\perp_1})$ is in $\star$. By Proposition 2.11, we have $\underline{\mathcal{H}} \simeq \operatorname{mod}\mathcal{C}$.

In the following example, we denote by "$\circ$" in a quiver the objects belong to a subcategory and by "$\cdot$" the objects do not.

Example 4.3. Let Λ be the path algebra of the following quiver

$$1 \longleftarrow 2 \longleftarrow 3 \longleftarrow 4 \longleftarrow 5$$

then we obtain the $\mathcal{E} = \operatorname{mod}\Lambda$.



Let

$$\mathcal{M} = \begin{array}{ccccccccc} \circ & & \cdot & & \cdot & & \circ & & \circ \\ & \circ & & \cdot & & \cdot & & \circ & \\ & & \circ & & \cdot & & \circ & & \\ & & & \circ & & \circ & & & \end{array}$$

Then $(\mathcal{M},\mathcal{M}^{\perp_1})$ is a cotorsion pair on $\mathcal{E}$ and the coheart $\mathcal{C} = {}^{\perp_1}\mathcal{M} \cap \mathcal{M} = {}^{\perp_1}\mathcal{M}$

$${}^{\perp_1}\mathcal{M} = \begin{array}{ccccccccc} \circ & & \cdot & & \cdot & & \cdot & & \cdot \\ & \circ & & \cdot & & \cdot & & \circ & \\ & & \circ & & \cdot & & \circ & & \\ & & & \circ & & \circ & & & \end{array} \qquad \mathcal{M}^{\perp_1} = \begin{array}{ccccccccc} \circ & & \cdot & & \cdot & & \cdot & & \circ \\ & \circ & & \cdot & & \cdot & & \circ & \\ & & \cdot & & \cdot & & \circ & & \\ & & & \circ & & \circ & & & \end{array}$$

We get $\mathcal{C}/\mathcal{P} = \operatorname{add}\left(\begin{smallmatrix}5\\4\\3\end{smallmatrix} \oplus \begin{smallmatrix}5\\4\end{smallmatrix}\right)$. And the heart is the following.

$$\underline{\mathcal{H}} = \begin{array}{ccccccccc} \cdot & & \circ & & \cdot & & \cdot & & \cdot \\ & \cdot & & \cdot & & \circ & & \cdot & \\ & & \cdot & & \circ & & \cdot & & \\ & & & \cdot & & \cdot & & & \end{array}$$

We can see that $\mathrm{mod}(\mathcal{C}/\mathcal{P}) \simeq \underline{\mathcal{H}}$.

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